Journal of Statistical Physics, Vol. 71, Nos. 5/6, 1993

Star–Triangle Relation for a Three-Dimensional Model

V. V. Bazhanov[1,2] **and R. J. Baxter**[1,3]

Received August 26, 1992

The solvable $sl(n)$-chiral Potts model can be interpreted as a three-dimensional lattice model with local interactions. To within a minor modification of the boundary conditions it is an Ising-type model on the body-centered cubic lattice with two- and three-spin interactions. The corresponding local Boltzmann weights obey a number of simple relations, including a restricted star–triangle relation, which is a modified version of the well-known star–triangle relation appearing in two-dimensional models. We show that these relations lead to remarkable symmetry properties of the Boltzmann weight function of an elementary cube of the lattice, related to the spatial symmetry group of the cubic lattice. These symmetry properties allow one to prove the commutativity of the row-to-row transfer matrices, bypassing the tetrahedron relation. The partition function per site for the infinite lattice is calculated exactly.

KEY WORDS: Three-dimensional solvable models; Zamolodchikov model; generalized chiral Potts model; symmetry relations; symmetry group of the cube; commuting transfer matrices; star–triangle relation; Yang–Baxter equation; star–star relation.

INTRODUCTION AND SUMMARY

There is a large number of solvable models in statistical mechanics where the bulk free energy and possibly other quantities such as order parameters and the correlation length can be calculated exactly. Mostly these models are two-dimensional, and only a few three-dimensional examples are known. The first such model was the two-dimensional Ising model, solved by

[1] Mathematics and Theoretical Physics, IAS, Australian National University, Canberra, ACT 0200, Australia.

[2] On leave of absence from the Institute for High Energy Physics, Protvino, Moscow Region, 142284, Russia.

[3] Isaac Newton Institute for Mathematical Sciences, Cambridge University, England.

0022-4715/93/0600-0839$07.00/0

Onsager in 1944.[1] It took some two decades to realize that two important ingredients of the Ising model—the star–triangle relation (STR) and the resulting commutativity of the transfer matrices—can be used to solve the other two-dimensional models.[2,3]

There is now quite a rich theory of solvable two-dimensional lattice models. Several different (but related) methods, mostly based on the commutativity of the transfer matrices, and thereby on the Yang–Baxter relation, have been developed (for review see refs. 4 and 5).

Can we find three-dimensional models with commuting transfer matrices? It is known that the tetrahedron relation[6] replaces the Yang–Baxter relation as a commutativity condition[7,8] for the three-dimensional cubic lattice. This relation contains thousands of distinct algebraic equations, and obviously it is very difficult to solve them. The only solution obtained by direct analysis of these equations is that of Zamolodchikov.[6,9] In fact, even in the simplest two-state spin case one has to solve 2^{14} equations instead of 2^6 in two dimensions. The symmetry properties could slightly reduce this number, but the jump in complexity is still enormous. Therefore one would like to find an alternative approach to the commutativity which would be based on simpler algebraic relations.

In the present paper we discuss one such scheme. We consider the solvable interaction-round-a-cube model[10] on the cubic lattice with N-valued spins ($N \geqslant 2$) at each site. To within a minor modification of the boundary condition it is equivalent to the $sl(n)$-chiral Potts model[11] (this model has an interesting history, which can be traced in refs. 12–17). On the other hand, it can be regarded as a multistate generalization of the Zamolodchikov model,[6] reducing to it when $N=2$. The Boltzmann weight function of eight corner spins around a cube has a very special form, such that by introducing an auxiliary center spin for each cube the model can be viewed as an Ising-type model on the body-centered cubic lattice with two- and three-spin interactions only. The corresponding Boltzmann weights are of course not arbitrary, and obey a number of simple relations. The most complicated among them is a "restricted star–triangle relation" (RSTR) which relates two- and three-spin weights. Remarkably enough this relation is a simplified version of the two-dimensional star–triangle relation (for the two-state spin case it can be obtained simply by specialization of rapidities in the STR of the 2D Ising model).

Here we show that this two-dimensional relation can be used in three dimensions so as to obtain a non-obvious rotational symmetry of the model. Hence, we derive the invariance properties of a Boltzmann weight function of the elementary cube of the lattice under transformations from the symmetry group of the cube. As in the Zamolodchikov model, this

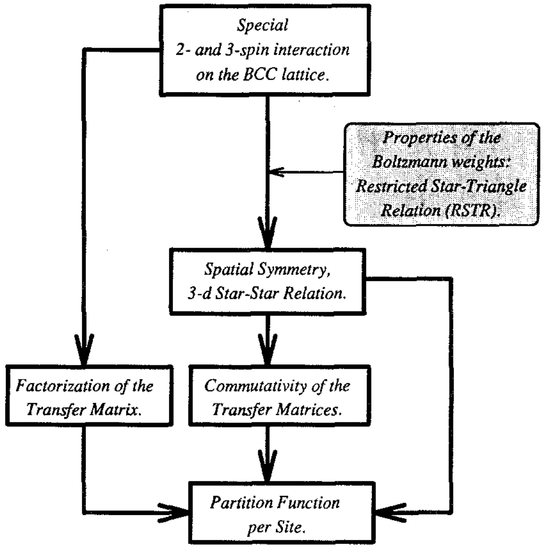

Fig. 1. The approach used in this paper.

Boltzmann weight function depends on three parameters, which can conveniently be chosen to be the dihedral angles $\theta_1, \theta_2, \theta_3$ between three "rapidity planes" passing through the cube. The symmetry properties are entirely consistent with such geometric interpretation of these angles.

The implication of the symmetry properties is twofold. First, they contain the three-dimensional star–star relation,[10,19] which results in the commutativity of the transfer matrices.[4] The other symmetry relations imply certain symmetry properties of the partition function per site for the lattice infinite in all three directions. These two features—commutativity and symmetry—allow us to find the partition function per site extending the $N=2$ Zamolodchikov case calculations of ref. 18 to the general N case. The result is quite surprising:

$$\log \kappa_N(\theta_1, \theta_2, \theta_3) = \frac{2(N-1)}{N} \log \kappa_2(\theta_1, \theta_2, \theta_3)$$

where κ_N is the partition function per site for the N-state model. In addition we show that there exists a three-dimensional free boson (Gaussian) model whose partition function per site is given by

$$\log \kappa_B(\theta_1, \theta_2, \theta_3) = -2 \log \kappa_2(\theta_1, \theta_2, \theta_3)$$

[4] Interestingly, this relation corresponds to the "central inversion" on the cubic lattice, or, equivalently, to the negation of all three of its axes.

This means that κ_N can be expressed as a (rational) power of some free boson determinant. It would be very interesting to understand the reason for this fact.

The sequence of our working is illustrated on Fig. 1. We hope this will assist the reader.

1. FORMULATION OF THE MODEL

1.1. The interaction-Round-a-Cube Model

Consider a simple cubic lattice $\mathscr{L}$ of M sites with periodic boundary conditions in each direction. At each site of $\mathscr{L}$ place a spin variable s taking $N \geqslant 2$ distinct values $s = 0, \ldots, N-1$, and allow all possible interactions of the spins within each elementary cube. The partition function reads

$$Z = \sum_{\text{spins}} \prod_{\text{cubes}} V(s_0 \,|\bar{s}_1, \bar{s}_2, \bar{s}_3 \,|\, s_1, s_2, s_3 |\, \bar{s}_0) \tag{1.1}$$

where $s_0, \ldots, \bar{s}_0$ are the eight spins of the cube arranged as in Fig. 2, and $V(s_0 | \bar{s}_1, \bar{s}_2, \bar{s}_3 | s_1, s_2, s_3 | \bar{s}_0)$ is the Boltzmann weight of the spin configuration $s_0, \ldots, \bar{s}_0$. Note that the spins $s_0, s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_0$ correspond, respectively, to a, b, c, d, e, f, g, h in Fig. 1 of ref. 10. The product is over all elementary cubes in $\mathscr{L}$.

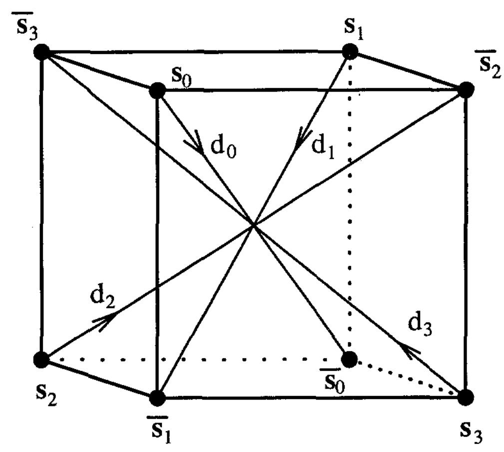

Fig. 2. Arrangements of the spins $s_0, \ldots, \bar{s}_0$ on the corner sites of an elementary cube of the simple cubic lattice $\mathscr{L}$; d_0, d_1, d_2, d_3 denote four oriented diagonals of the cube.

Taking the lattice to have m horizontal layers and letting ϕ_i denote all spins in layer i, one can rewrite (1.1) as

$$Z = \sum_{\phi_1} \sum_{\phi_2} \cdots \sum_{\phi_m} T_{\phi_1 \phi_2} T_{\phi_2 \phi_3} \cdots T_{\phi_m \phi_1} = \operatorname{Tr} T^m \tag{1.2}$$

where T is a layer-to-layer transfer matrix whose elements are the products of all the V functions of cubes between two adjacent layers. Clearly, T depends on the Boltzmann weight function V, so we can write it as $T(V)$.

Obviously, we cannot solve the model (1.1) for an arbitrary Boltzmann weight function V. What we apparently can do is to solve a restricted class of models whose transfer matrices form commuting families, i.e., such that any two transfer matrices $T(V)$ and $T(V')$ belonging to the same family commute,

$$[T(V), T(V')] = 0 \tag{1.3}$$

In all known cases this commutativity condition can always be reformulated as a relation for the local Boltzmann weight functions V and V'. In two dimensions the corresponding relation is well known: it is the Yang–Baxter equation for the local Boltzmann weights. This equation involves three different Boltzmann weight functions. A straightforward generalization[7,8] of this construction for a three-dimensional lattice leads to the tetrahedron equation,[6] involving four different Boltzmann weight functions. Even in the simplest two-valued spin case this equation involves thousands of distinct relations and obviously is very difficult to solve. The only known solution obtained from the direct analysis of these relations is Zamolodchikov's.[6,9]

In this paper we present a somewhat different approach to the commutativity in three dimensions, which could, however, be applied to the Zamolodchikov model as well. We consider the interaction-round-a-cube model of ref. 10 where the Boltzmann weight function of eight corner spins around a cube, V, has a very special form, such that by introducing an auxiliary spin for each cube the model can be viewed as an Ising-type model on the body-centered cubic lattice with two- and three-spin interactions only. The corresponding two-spin and three-spin Boltzmann weights obey a number of simple relations, including "a restricted star–triangle relation" (RSTR), which allows us to prove the commutativity of the transfer matrices and calculate the partition function per site by the use of the symmetry properties of the model.

Before ending this subsection, we note that there are simple transformations of V that do not change (1.1). For instance, if we multiply $V(s_0, \ldots, \bar{s})$ by $F(s_0, \bar{s}_2, s_1, \bar{s}_3)/F(\bar{s}_1, s_3, \bar{s}_0, s_2)$, then each horizontal face of

$\mathscr{L}$ acquires an F factor from the cube below it and a canceling $1/F$ factor from the cube above. The effect on the transfer matrix T is to apply a diagonal similarity transformation. Provided F is the same for $T(V)$ and $T(V')$, the commutation relation is unaffected. This is an example of a "face-factor" transformation. Similarly, one could apply "edge-factor" and "site-factor" transformations that leave (1.1) and (1.3) unchanged.

1.2. The Boltzmann Weights

Let a, b, c be integers and v be a complex variable. Introduce the notation

$$\omega = \exp(2\pi i/N), \qquad \omega^{1/2} = \exp(i\pi/N) \tag{1.4}$$

$$\gamma(a, b) = \omega^{ab}, \qquad \Phi(a) = (\omega^{1/2})^{a(N+a)} \tag{1.5}$$

$$\Delta(v) = (1 - v^N)^{1/N} \tag{1.6}$$

Note that

$$\begin{gathered} \gamma(a, b) = \gamma(b, a) = \gamma(a + N, b) \\ \Phi(a + N) = \Phi(a) \end{gathered} \tag{1.7}$$

Define a function $w(v, a)$ such that

$$\frac{w(v, a)}{w(v, 0)} = [\Delta(v)]^a \prod_{j=1}^{a} (1 - \omega^j v)^{-1} \tag{1.8}$$

Obviously, the LHS of this equation is a multivalued function of v, but it is a singlevalued function of a point $(v, \Delta(v))$ on the algebraic curve

$$v^N + \Delta^N = 1 \tag{1.9}$$

Below we shall interpret w in this latter way, suppressing, however, an explicit dependence on the phase of $\Delta(v)$ in the arguments of w.

Now, fix four complex parameters p, p', q, q', and define

$$v_1 = q'/(\omega p'), \qquad v_2 = q'/p, \qquad v_3 = p/q, \qquad v_4 = p'/q \tag{1.10}$$

Obviously

$$\omega v_1 v_4 = v_2 v_3 \tag{1.11}$$

The Boltzmann weight function of the model is given by Eqs. (2.9) and (2.10) of ref. 10. Taking into account Eq. (1.18) (given below) and omitting

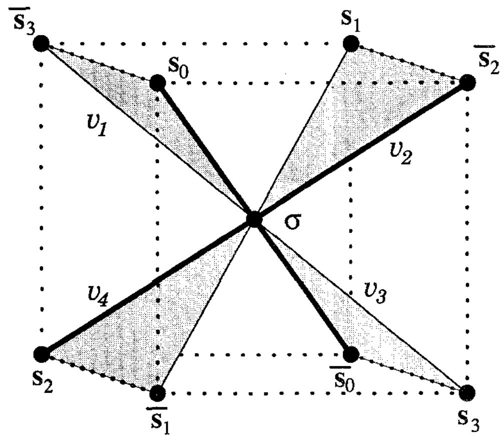

Fig. 3. A typical elementary cube of $\mathscr{L}$, with corner spins $s_0,..., \bar{s}_0$ and the center spin σ.

equivalence transformation factors (which do not change the partition function), one can write it in the form

$$
\begin{aligned}
&V(s_0 \,|\bar{s}_1, \bar{s}_2, \bar{s}_3 \,|\, s_1, s_2, s_3 |\, \bar{s}_0 \,\|\, v_1, v_2, v_3, v_4) \\
&\quad = \rho \sum_{\sigma=0}^{N-1} \frac{w(v_2, s_1 - \bar{s}_2 + \sigma)\, w(v_3, s_3 - \bar{s}_0 - \sigma)\, \gamma(s_0, \sigma)\, \gamma(\bar{s}_0, \sigma)}{w(v_1, \bar{s}_3 - s_0 + \sigma)\, w(v_4, \bar{s}_1 - s_2 - \sigma)\, \gamma(s_2, \sigma)\, \gamma(\bar{s}_2, \sigma)}
\end{aligned}
\tag{1.12}
$$

where ρ is a normalization factor depending on $v_1,..., v_4$. Note that the Boltzmann weight function (1.12) describes a very special type of interaction of eight spins around the cube. A typical cube with its center spin σ is shown in Fig. 3. There are three-spin interactions on the shaded triangles, described by $w(v, a)$ or by $1/w(v, a)$ in (1.12). There are also two-spin interactions such as $\gamma(s_0, \sigma)$ or $1/\gamma(s_2, \sigma)$ associated with the edges linking σ to $s_0, \bar{s}_0, s_2, \bar{s}_2$ (these edges are denoted by heavy lines in Fig. 3). In addition, the three-spin and two-spin interaction weights w and s obey a number of important relations which we now discuss. These relations, rather than the explicit form of w and γ given by (1.5) and (1.8), enable us to execute the program stated in the introduction and calculate the partition function per site in the thermodynamic limit.

1.3. Restricted Star–Triangle Relation

Consider the following automorphism of the curve (1.9):

$$(v, \Delta(v)) \to (\tilde{v}, \Delta(\tilde{v})) \tag{1.13}$$

$$\tilde{v} = \frac{1}{wv}, \qquad \Delta(\tilde{v}) = \frac{w^{-1/2}\Delta(v)}{v} \tag{1.14}$$

One can easily show that $\gamma(a, b)$, $\Phi(a)$, and $w(v, a)$ satisfy the following properties:

$$\gamma(a, b+c) = \gamma(a, b)\,\gamma(a, c) \tag{1.15}$$

$$\sum_{b=0}^{N-1} \gamma(a, b)\,\gamma(-b, c) = N\delta_{ac} \tag{1.16}$$

$$\Phi(a+b) = \Phi(a)\,\Phi(b)\,\gamma(a, b) \tag{1.17}$$

$$\frac{w(v, a)\,w(\tilde{v}, -a)}{w(v, 0)\,w(\tilde{v}, 0)} = \Phi^{-1}(a) \tag{1.18}$$

where $\tilde{v}$ is related to v by Eq. (1.14). Moreover, these functions satisfy two less trivial relations. For the moment let v_1, v_2, v_3, v_4 denote four arbitrary complex variables; then

$$\begin{aligned} F_1(v_1, v_2 \mid a, b) &\equiv \sum_{l=0}^{N-1} \frac{w(v_2, a-l)}{w(v_1, -l)\,\gamma(b, l)} \\ &= \varphi_1(v_1, v_2)\,\frac{w(v_2', -b)\,w(v_2/(\omega v_1), a)}{w(v_1', a-b)} \end{aligned} \tag{1.19}$$

$$\begin{aligned} F_2(v_3, v_4 \mid a, b) &\equiv \sum_{l=0}^{N-1} \frac{w(v_3, -l)\,\gamma(b, l)}{w(v_4, a-l)} \\ &= \varphi_2(v_3, v_4)\,\frac{w(v_3', a-b)}{w(v_4', -b)\,w(v_4/v_3, a)} \end{aligned} \tag{1.20}$$

where φ_1 and φ_2 are scalar functions and

$$v_1' = \frac{v_2 \Delta(v_1)}{\omega v_1 \Delta(v_2)}, \qquad \Delta(v_1') = \frac{\Delta(v_2/\omega v_1)}{\Delta(v_2)} \tag{1.21a}$$

$$v_2' = \frac{\Delta(v_1)}{\Delta(v_2)}, \qquad \Delta(v_2') = \frac{v_2 \Delta(v_2/\omega v_1)}{\Delta(v_2)} \tag{1.21b}$$

$$v_3' = \frac{v_4 \Delta(v_3)}{v_3 \Delta(v_4)}, \qquad \Delta(v_3') = \frac{\Delta(v_4/v_3)}{\Delta(v_4)} \tag{1.21c}$$

$$v_4' = \frac{\Delta(v_3)}{\omega \Delta(v_4)}, \qquad \Delta(v_4') = \frac{v_3 \Delta(v_4/v_3)}{\Delta(v_4)} \tag{1.21d}$$

Each of the relations (1.19), (1.20) is a corollary of another and the properties (1.15)–(1.18). The phases of $\Delta(v_4/v_3)$ and $\Delta(v_2/\omega v_1)$ can be chosen arbitrarily, since they cancel out the RHS of (1.19) and (1.20).

The reader may have noticed that relation (1.19) (or (1.20)) is a particular case of the usual star–triangle relation. Using (1.17) and (1.18) for $w(v_1, -l)$ and $w(v_2', -b)$ in (1.19) and then replacing a, b, l there by $a-c$, $b-c$, $l-c$ respectively one obtains

$$\sum_{l=0}^{N-1} w(v_2, a-l)\, w(1/(\omega v_1), l-c)\, \Phi(b-l)$$

$$= \varphi_1'(v_1, v_2) \frac{w(v_2/(\omega v_1), a-c)}{w(v_1', a-b)\, w(1/(\omega v_2'), b-c)} \tag{1.22}$$

where φ_1' is another scalar factor simply related with φ_1. We see that the LHS of this equation is the sum of the product of the three functions (each depending on the difference of two spins), while the RHS is the product of three such functions. Unlike the usual star–triangle relation, there is some asymmetry in the LHS of (1.19): the function γ does not depend on any continuous parameters. It is quite possible that (1.19) is a particular case of a more general relation and γ is just a limiting value of a more complex function. In fact, this is exactly so for $N=2$ when (1.19) and (1.20) can be obtained by a specialization of rapidities in the star–triangle relation of the two-dimensional Ising model.[1] This is why we call (1.19), (1.20) the "restricted star–triangle relations."

2. THE SYMMETRY RELATIONS

2.1. The Cube Symmetry

Consider the cube $\mathscr{C}$ shown in Fig. 2. The eight spins $s_0, \ldots, \bar{s}_0$ at the corners of $\mathscr{C}$ can be grouped into four ordered pairs:

$$d_j = (s_j, \bar{s}_j), \qquad j = 0, \ldots, 3 \tag{2.1}$$

corresponding to four diagonals $d_0, \ldots, d_j$, which we assume to be oriented and directed from s_j to $\bar{s}_j$, $j=0, \ldots, 3$. The spatial symmetry group of the cube $\mathscr{G}(\mathscr{C})$ (consisting of all possible reflection and rotations which map the cube to itself) has a structure of the direct product

$$\mathscr{G}(\mathscr{C}) = C_2 \times S_4 \tag{2.2}$$

of the cyclic group of order 2, C_2, generated by the central inversion P (which reverses the directions of all four diagonals $d_0, \ldots, d_3$) and the symmetric group S_4 of order 24, consisting of the transformations which permute the diagonals preserving their directions.[5]

Let

$$s_{0123}=(s_0\,|\bar{s}_1,\bar{s}_2,\bar{s}_3\,|\,s_1,s_2,s_3|\,\bar{s}_0) \tag{2.3}$$

denote the sequence of the spins $s_0,\ldots,\bar{s}_0$, corresponding to their basic arrangement at the corners of $\mathscr{C}$ as shown in Fig. 2.

Obviously, the transformations from $\mathscr{G}(\mathscr{C})$ induce some permutations of the spins $s_0,\ldots,\bar{s}_0$. According to the above discussion of the structure (2.2) of $\mathscr{G}(\mathscr{C})$ these transformations map the sequence (2.3) either to s_{ijkl} or to $\bar{s}_{ijkl}$,

$$s_{ijkl}=(s_i\,|\bar{s}_j,\bar{s}_k,\bar{s}_l\,|\,s_j,s_k,s_l|\,\bar{s}) \tag{2.4}$$

$$\bar{s}_{ijkl}=(\bar{s}_i\,|s_j,s_k,s_l\,|\,\bar{s}_j,\bar{s}_k,\bar{s}_l|\,s) \tag{2.5}$$

where (i, j, k, l) is some permutation of (0, 1, 2. 3).

Consider two elements of $\mathscr{G}(\mathscr{C})$, specifying their action on the spins. Let (i, j, k, l) be any permutation of (0, 1, 2, 3), and let $R\in\mathscr{G}(\mathscr{C})$,

$$Rs_{ijkl}=\bar{s}_{lkij},\qquad R\bar{s}_{ijkl}=s_{lkij},\qquad R^4=1 \tag{2.6}$$

denote the 90° rotation around the axes passing through the centers of the top and bottom faces of $\mathscr{C}$, while $T\in\mathscr{G}(\mathscr{C})$,

$$Ts_{ijkl}=s_{ikjl},\qquad T\bar{s}_{ijkl}=\bar{s}_{ikjl},\qquad T^2=1 \tag{2.7}$$

denotes the reflection with respect to the plane passing through the corners of $\mathscr{C}$ occupied by the spins $s_0, s_3, \bar{s}_0, \bar{s}_3$ in Fig. 2.

One could easily check that these two elements R and T generate the whole group $\mathscr{G}(\mathscr{C})$. In particular, the central inversion P,

$$Ps_{ijkl}=\bar{s}_{ijkl},\qquad P\bar{s}_{ijkl}=s_{ijkl},\qquad P^2=1 \tag{2.8}$$

can be expressed as

$$P=(RT)^3 \tag{2.9}$$

2.2. The Angle Parametrization

Let θ_1, θ_2, θ_3 denote angles of a spherical triangle and a_1, a_2, a_3 denote three sides of this triangle opposite to the angles $\theta_1, \theta_2, \theta_3$. Define the related variables

$$\begin{aligned}\alpha_0&=(\theta_1+\theta_2+\theta_3-\pi)/2, &\qquad \alpha_i&=\theta_i-\alpha_0\\ \beta_0&=(2\pi-a_1-a_2-a_3)/2, &\qquad \beta_i&=\pi-\beta_0-a_i\end{aligned} \tag{2.10}$$

[5] For the even permutation the corresponding transformation is a pure rotation of the cube, while for the odd permutation it is a rotation followed by the central inversion P.

for $i=1, 2, 3$. Choose $\theta_1, \theta_2, \theta_3$ so that $\theta_1, \theta_2, \theta_3, \alpha_0,..., \alpha_3, \beta_0,..., \beta_3$ are all real, between 0 and π. Further, define (taking real, positive values of roots)

$$\begin{aligned} S_i &= [\sin(\theta_i/2)]^{1/N}, \qquad C_i = [\cos(\theta_i/2)]^{1/N} \\ T_i &= [\tan(\theta_i/2)]^{1/N}, \qquad z_i = \exp(ia_i/N) \end{aligned} \tag{2.11}$$

for $i=1, 2, 3$ and

$$u_i = \exp(i\beta_i/N), \qquad c_i = [\cos(\alpha_i/2)]^{1/2}, \qquad i=0,..., 3 \tag{2.12}$$

Now parametrize p, p', q, q' in (1.10) as follows[6]:

$$p = z_3^{-1} T_1, \quad p' = \omega^{-1/2} z_3^{-1} T_1^{-1}, \quad q = T_2^{-1}, \qquad q' = \omega^{-1/2} T_2$$

Then

$$\begin{aligned} v_1 &= \omega^{-1} z_3 T_1 T_2, \qquad v_2 = \omega^{-1/2} z_3 T_2/T_1 \\ v_3 &= z_3^{-1} T_1 T_2, \qquad v_4 = \omega^{-1/2} z_3^{-1} T_2/T_1 \end{aligned} \tag{2.13}$$

Now choose the phases of $\Delta(v_1),..., \Delta(v_4)$ such that

$$\begin{aligned} \Delta(v_1) &= S_3/C_1 C_2 u_3, \qquad \Delta(v_2) = C_3 u_1/S_1 C_2 \\ \Delta(v_3) &= S_3 u_3/C_1 C_2, \qquad \Delta(v_4) = C_3/S_1 C_2 u_1 \end{aligned} \tag{2.14}$$

$$\Delta(v_4/v_3) = \Delta(v_2/\omega v_1) = S_1^{-2} \tag{2.15}$$

With these definitions the Boltzmann weight function (1.12) can be regarded as a function of the three independent variables $\theta_1, \theta_2, \theta_3$ or, equivalently, of the four dependent variables $\alpha_0,..., \alpha_3$ constrained by the relation

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \pi \tag{2.16}$$

so we can write the LHS of (1.12) as $V(s_0,..., \bar{s}_0 \,\|\, \alpha_0,..., \alpha_3)$.

Note that the angles $\theta_1, \theta_2, \theta_3$ can be viewed as the dihedral angles between three "rapidity planes" passing through the cube exactly as in the Zamolodchikov model.

2.3. The Normalization of the Weights

Now we fix the normalization factors $w(v, 0)$ in (1.8) and ρ in (1.12). First let us choose $w(v, 0)$ such that

$$\prod_{a=0}^{N-1} w(v, a) = 1 \tag{2.17}$$

[6] This parametrization has been obtained by a formal generalization of Eqs. (4.19) and (4.24) of ref. 10 for arbitrary values of N.

With this normalization define

$$D_+(v) = (\det_N \|w(v, a-b)\|)^{1/N}$$
$$D_-(v) = (\det_N \|1/w(v, a-b)\|)^{1/N} \tag{2.18}$$

Also, set

$$S_+ = (\det_N \|\gamma(a, b)\|)^{1/N}$$
$$S_- = (\det_N \|1/\gamma(a, b)\|)^{1/N} \tag{2.19}$$

Regarding a, b in (1.19), (1.20) as matrix indices running over the values $0,..., N-1$ and taking the determinants of both sides of these equations, one gets, by using (2.17)–(2.19),

$$\varphi_1(v_1, v_2) = D_+(v_2)\, S_-/D_-(v_1')$$
$$\varphi_2(v_3, v_4) = D_-(v_4)\, S_+/D_+(v_3') \tag{2.20}$$

where v_1', v_3' are given by (1.21). Explicit calculations with Eqs. (1.8) give

$$D_\pm(v) = c_\pm [v/\Delta(v)]^{(N-1)/2}$$
$$S_+ S_- = N \tag{2.21}$$

where $c_\pm$ are (inessential) constants. Note, in particular, that when $v_1,..., v_4$ are parametrized by (2.13)–(2.15) we have

$$\phi_1(v_1, v_2)\, \phi_2(v_3, v_4) = N\left(\frac{\sin\theta_2}{\sin\theta_3}\right)^{(N-1)/N} \tag{2.22}$$

Further, set the normalization factor in (1.12) as

$$\rho = (2\xi)^{2(N-1)/N}/N \tag{2.23}$$

$$2\xi = (\tfrac{1}{2}\sin\theta_3)^{1/2}/c_0 c_1 c_2 c_3 \tag{2.24}$$

where $c_0,..., c_3$ are given by (2.12).

We want to calculate the free energy, or equivalently the partition function per site

$$\kappa = Z^{1/M} \tag{2.25}$$

Note that when $\alpha_0 = \alpha_2 = 0$ we have from (2.10)–(2.13)

$$v_1 = v_2 = v_3 = v_4 = 0, \qquad 2\xi = 1 \tag{2.26}$$

while from (1.8)

$$w(0, a) = w(0, 0), \quad \forall a \tag{2.27}$$

Substituting (2.26), (2.27) into (1.12) and using (1.15), (1.16), one gets

$$V = \delta_{s_0 - \bar{s}_2, s_2 - \bar{s}_0} \tag{2.28}$$

Ignoring irrelevant boundary contributions, it follows that for M large

$$\kappa = 1 \qquad \text{when} \quad \alpha_0 = \alpha_2 = 0 \tag{2.29}$$

2.4. The Symmetry Properties of the Boltzmann Weights

The definition (1.12) of the Boltzmann weight function V is obviously rather asymmetric with respect to the orientation of the elementary cube (see Fig. 3). We shall see, however, that despite this visual asymmetry, the Boltzmann weight function (1.12) has a remarkable hidden symmetry. Below we shall show that up to the equivalence transformation factors, which do not affect the partition function, the Boltzmann weight function remains unchanged upon the permutations of the corner spins $s_0, \ldots, \bar{s}_0$ induced by symmetry transformations of the cube complemented by the corresponding transformation of the variables v_1, v_2, v_3, v_4 (or, conveniently, the variables $\alpha_0, \ldots, \alpha_3$). Obviously, it is enough to prove this just for two transformations (2.6) and (2.7), since they generate the whole symmetry group of the cube.

From (1.3), (1.18), and (2.17) it follows that

$$w(v, 0)\, w(\tilde{v}, 0) = e^{i\pi(N^2 - 1)/6} \tag{2.30}$$

is a constant independent of v. Let us interchange α_1 with α_2 leaving α_0, α_3 intact. From (2.13) this is equivalent to the replacement of v_1, v_2, v_3, v_4 by v_1, $\tilde{v}_4$, v_3, $\tilde{v}_2$, respectively. Using (2.30) and noting that (2.23) remains unchanged, one can easily trace the effect of this interchange on the weight function (1.12),

$$V(s_{0123} \,\|\, \alpha_0, \alpha_1, \alpha_2, \alpha_3) = \frac{\Phi(s_2 - \bar{s}_1)}{\Phi(s_1 - \bar{s}_2)}\, V(s_{0213} \,\|\, \alpha_0, \alpha_2, \alpha_1, \alpha_3) \tag{2.31}$$

where we have used the short notations (2.4) for the spin configuration of the corner spins. This gives the transformation law of V under the reflection T, (2.7).

Further, using (1.15), (1.16), rewrite (1.12) in the form

$$\frac{\gamma(s_0-\bar{s}_3, s_0-\bar{s}_2)\,\rho}{\gamma(\bar{s}_0-s_3, \bar{s}_0-s_2)\,N} \sum_{\mu=0}^{N-1} \left\{ \frac{\gamma(s_3+\bar{s}_3, \mu)}{\gamma(s_0+\bar{s}_0, \mu)} \right.$$
$$\times F_1(v_1, v_2 \mid s_0+s_1-\bar{s}_2-\bar{s}_3, s_0-\bar{s}_2-\mu)$$
$$\left. \times F_2(v_3, v_4 \mid \bar{s}_0+\bar{s}_1-s_2-s_3, \bar{s}_0-s_2+\mu) \right\} \tag{2.32}$$

where we have used the first "star" forms of F_1, F_2 given in (1.19) and (1.20). In particular, in F_1 we have replaced a, b, l of (1.19) by $s_0+s_1-\bar{s}_2-\bar{s}_3$, $s_0-\bar{s}_2-\mu$ and $s_0-\bar{s}_3-\sigma$ respectively. Now we replace the star forms by the equivalent "triangular" forms, i.e. we use the identities (1.19) and (1.20). Writing the identity for F_1 in full, it is

$$\gamma(s_0-\bar{s}_2-\mu, s_0-\bar{s}_3)\, F_1(v_1, v_2 \mid s_0+s_1-\bar{s}_2-\bar{s}_3, s_0-\bar{s}_2-\mu)$$
$$= \sum_{\sigma=0}^{N-1} \frac{w(v_2, s_1-\bar{s}_2+\sigma)\,\gamma(s_0-\bar{s}_2-\mu, \sigma)}{w(v_1, \bar{s}_3-s_0+\sigma)}$$
$$= \varphi(v_1, v_2)\,\gamma(s_0-\bar{s}_2, s_0-\bar{s}_3)\, w(v_2/(\omega v_1), s_0+s_1-\bar{s}_2-\bar{s}_3)$$
$$\times \left\{ \frac{w(v_2', \bar{s}_2-s_0+\mu)\,\gamma(\bar{s}_3, \mu)}{w(v_1', s_1-\bar{s}_3+\mu)\,\gamma(s_0, \mu)} \right\} \tag{2.33}$$

where v_1', v_2' are defined by (1.21). This identity (2.33) although obtained from the planar (restricted) star–triangle relation (1.19), has a natural three-dimensional interpretation, as is shown in Fig. 4. The "star" (the second line in (2.33)) now involves the three-spin interactions on the shaded triangles and the two-spin interaction on the dashed and heavy lines in Fig. 4a, while the "triangle" (the third line in (2.33)) involves two-spin interactions on the heavy and dashed lines, three-spin interactions on two shaded triangles and the four-spin interaction on the upper square shown in Fig. 4b.

The application of (1.20) to F_2 has the similar graphical interpretation for the lower half of the cube. Further, for the complete lattice the interactions introduced on the upper face in Fig. 4b (given by the w and γ factors before the curly braces in the third line of (2.33)) are canceled by the corresponding contributions from the surrounding cubes (particularly the one above). Thus the main effect of these identities is to rotate each cube through 90° about the vertical axis, replacing $v_1,..., v_4$ by $v_1',..., v_4'$. This is the basic trick we use in the paper.

Further, using (1.21) and (2.10)–(2.14) we find that replacing $v_1,..., v_4$ by $v_1',..., v_4'$ is equivalent to replacing $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ by $\alpha_3, \alpha_1, \alpha_0, \alpha_1$ and,

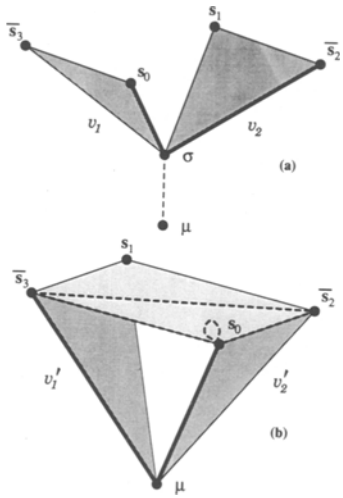

Fig. 4. A three-dimensional interpretation of the "star-triangle relation" (2.33).

finally, writing the normalization factor ρ in (2.32) (given by (2.23)) as $\rho(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and collecting all the other scalar factors there we get for their product

$$N^{-1}\rho(\alpha_0, \alpha_1, \alpha_2, \alpha_3)\, \varphi_1(v_1, v_2)\, \varphi_2(v_3, v_4)$$

which with account of (2.22), (2.24) is exactly equal to $\rho(\alpha_3, \alpha_1, \alpha_0, \alpha_1)$. Hence we obtain

$$\begin{aligned} &V(s_{0123} \,\|\, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \\ &\quad = \frac{\gamma(s_0 - \bar{s}_3, s_0 - \bar{s}_2)}{\gamma(\bar{s}_0 - s_3, \bar{s}_0 - s_2)} \frac{w(v_4/v_3, s_0 + s_1 - \bar{s}_2 - \bar{s}_3)}{w(v_4/v_3, \bar{s}_0 + \bar{s}_1 - s_2 - s_3)} \\ &\quad\quad \times V(\bar{s}_{3201} \,\|\, \alpha_3, \alpha_2, a_0, \alpha_1) \end{aligned} \tag{2.34}$$

which gives the required transformation of V under the rotation R, (2.6). Combining now Eqs. (2.32) and (2.34) and using (1.18), (2.30) for the functions w in (2.34), one gets the relation corresponding to the RT transformation,

$$\begin{aligned} &V(s_{0123} \,\|\, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \\ &\quad = \frac{\Phi(s_0)\, \Phi(s_3)\, \gamma(\bar{s}_2, \bar{s}_0 - s_3)\, w(v_2 v_3, s_1 + s_3 - \bar{s}_0 - \bar{s}_2)}{\Phi(\bar{s}_0)\, \Phi(\bar{s}_3)\, \gamma(s_2, s_0 - \bar{s}_3)\, w(v_2 v_3, \bar{s}_1 + \bar{s}_3 - s_0 - s_2)} \\ &\quad\quad \times V(\bar{s}_{3102} \,\|\, \alpha_3, \alpha_1, \alpha_0, \alpha_2) \end{aligned} \tag{2.35}$$

where the argument of w's can be written as $v_2 v_3 = q'/q = \omega^{-1/2} T_2^2$.

Remembering Eq. (2.9) and iterating (2.35) three times, one obtains

$$\frac{V(s_{0123}\,\|\,\alpha_0,\alpha_1,\alpha_2,\alpha_3)}{V(\bar{s}_{0123}\,\|\,\alpha_0,\alpha_1,\alpha_2,\alpha_3)}$$
$$=\frac{\Phi^2(s_0)}{\Phi^2(\bar{s}_0)}\frac{\gamma(\bar{s}_2,\bar{s}_0-s_3)}{\gamma(s_2,s_0-\bar{s}_3)}\frac{\gamma(s_0,s_3-\bar{s}_2)}{\gamma(\bar{s}_0,\bar{s}_3-s_2)}\frac{\gamma(\bar{s}_3,\bar{s}_2-s_0)}{\gamma(s_3,s_2-\bar{s}_0)}$$
$$\times\frac{w(v_2v_3,s_1+s_3-\bar{s}_0-\bar{s}_2)}{w(v_2v_3,\bar{s}_1+\bar{s}_3-s_0-s_2)}$$
$$\times\frac{w(v_5,\bar{s}_1+\bar{s}_2-s_0-s_3)}{w(v_5,s_1+s_2-\bar{s}_0-\bar{s}_3)}\frac{w(v_4/v_3,s_0+s_1-\bar{s}_2-\bar{s}_3)}{w(v_4/v_3,\bar{s}_0+\bar{s}_1-s_2-s_3)} \tag{2.36}$$

where

$$v_5=\frac{v_4\Delta(v_1)\,\Delta(v_3)}{v_3\Delta(v_2)\,\Delta(v_4)}=\omega^{-1/2}T_3^2, \qquad \Delta(v_5)=C_3^{-2} \tag{2.37}$$

One can check that (2.36) is exactly the three-dimensional star–star relation conjectured previously [Eq. (6.1) of ref. 10].

As we remarked before, the angles θ_1, θ_2, θ_3 can be interpreted as the dihedral angles between the three rapidity planes rigidly connected with the cube. Then, from geometric considerations these angles should be very simply transformed by the cube symmetry group $\mathscr{G}(\mathscr{C})$. Namely, the related variables α_0, α_1, α_2, α_3 given by (2.10) should just permute for any transformation from $\mathscr{G}(\mathscr{C})$. This is entitely consistent with (2.31), (2.34).

3. PARTITION FUNCTION

3.1. Factorization and Commutativity

Consider two successive layers of $\mathscr{L}$ with l spins ϕ on the lower layer, ϕ' on the upper. In the center of each intervening cube we have the central spin σ as in Fig. 3. Let ϕ'' denote the set of all these σ-spins between ϕ and ϕ''. Then, because the top spins s_0, s_1, $\bar{s}_2$, $\bar{s}_3$ in Fig. 3 interact only with one another and with σ, and similarly for the bottom spins, we can write the elements of the transfer matrix T as

$$T_{\phi,\phi'}=\rho^l\sum_{\phi''}X_{\phi,\phi''}Y_{\phi'',\phi'} \tag{3.1}$$

where

$$X_{\phi,\phi''} = \prod_{\text{cubes}} \frac{w(v_3, s_3 - \bar{s}_0 - \sigma)\, \gamma(\bar{s}_0, \sigma)}{w(v_4, \bar{s}_1 - s_2 - \sigma)\, \gamma(s_2, \sigma)}$$
$$Y_{\phi'',\phi} = \prod_{\text{cubes}} \frac{w(v_2, s_1 - \bar{s}_2 + \sigma)\, \gamma(\bar{s}_0, \sigma)}{w(v_1, \bar{s}_3 - s_0 + \sigma)\, \gamma(s_2, \sigma)} \tag{3.2}$$

The products are over all l cubes between the two layers; for each cube $s_0, s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_0$ are the eight spins shown in Fig. 3.

Obviously we can regard $X_{\phi,\phi''}$ as the element of a matrix X. From (3.2) this matrix depends on v_3 and v_4, so it can be written as $X(v_3, v_4)$. With similar conventions for Y, (3.1) implies

$$T = \rho^l X(v_3, v_4)\, Y(v_1, v_2) \tag{3.3}$$

Thus $\rho^{-l}T$ factors into a product of two matrices, one dependent on $\theta_1, \theta_2, \theta_3$ only via v_3 and v_4, the other via v_1 and v_2.

In ref. 10 it was shown that transfer matrices T form two-parameter commuting families. More precisely, when parametrizing p, p', q, q' through the angles (2.13), Eqs. (1.5), (2.11), (2.17), and (2.22) of ref. 10 imply that two transfer matrices (with different values of $\theta_1, \theta_2, \theta_3$) commute provided they have the same value of θ_1,

$$[T(\theta_1, \theta_2, \theta_3), T(\theta_1, \theta_2', \theta_3')] = 0, \qquad \forall\, \theta_2, \theta_3, \theta_2', \theta_3' \tag{3.4}$$

The proof was based on the Yang–Baxter equation for the $sl(n)$-chiral Potts model.[11,17,19] Alternatively, we can establish the commutativity property (3.4) directly from the three-dimensional star–star relation (2.36) not referring to those results. This will be done below in this section.

If θ_1 is known, then v_2 can be determined from v_1 and v_4 from v_3 by corollaries of (2.13):

$$v_2 = \omega^{1/2} \left(\cot \frac{\theta_1}{2}\right)^{2/N} v_1$$
$$v_4 = \omega^{-1/2} \left(\cot \frac{\theta_1}{2}\right)^{2/N} v_3 \tag{3.5}$$

The next step is to show that the factorization property (3.3) remains true when the matrices T, X, Y are appropriately diagonalized. To ensure this, it is necessary to slightly modify the model.

There are two sorts of vertical faces in $\mathscr{L}$: those whose perpendiculars run in front-to-back direction (such as $s_0\bar{s}_2 s_3 \bar{s}_1$ and $\bar{s}_3 s_1 \bar{s}_0 s_2$ in Fig. 3), and those whose perpendiculars run right-to-left. Call the former type FB, the latter RL. At the center of each FB face place a spin μ, with values $0,..., N-1$. Let the spins on the front and back faces in Fig. 3 be μ and μ', respectively. Choose them so that

$$\sigma = \mu - \mu' \pmod N \tag{3.6}$$

Do this for all cubes in $\mathscr{L}$. If σ' is the spin behind σ, and σ'' is the spin behind that, etc. then, on using the cyclic boundary conditions, we observe that

$$\sigma + \sigma' + \sigma'' + \cdots = (\mu - \mu') + (\mu' - \mu'') + (\mu'' - \mu''') + \cdots = 0 \pmod N \tag{3.7}$$

[Each μ-spin occurs twice with the opposite signs. If $\mathscr{L}$ has n layers perpendicular to the gront-to-back direction, then there are n σ-spins on the LHS of (3.6).] Thus we can use (3.6) only if the sum of each horizontal front-to-back line of σ-spins is constrained to be zero. This is merely a change of boundary conditions, and in the limit of n large it should have no effect on the partition function per site κ. We shall refer to the model subject to these constraints as the modified model.

For the modified model, (3.1) and (3.2) formally remains the same, but ϕ'' is now the set of all μ-spins and σ in (3.2) is now given by (3.6). Note also that in this case the matrices X, Y, T are unchanged under overall shifts of all μ-spins, or all s-spins, on any front-to-back line. Therefore X, Y, and T have nonzero entries only in the diagonal block with respect to the subspace invariant under all such shifts.

For the moment, let us ignore the constraints (3.6), and work with the original model. The elements of the transfer matrix XY in (3.1)–(3.3) are obtained by taking the product over all cubes in a layer of the function $V(s_{0123} \| \alpha_0,..., \alpha_3)$ given by (1.12), (2.3). Let us replace the function V by $\bar{V} = V(\bar{s}_{0123} \| \alpha_0,..., \alpha_3)$ with $\bar{s}_{0123}$ given by (2.5) and denote the corresponding transfer matrix as $\bar{T}$. Then with the same arguments which led to (3.3) one can show that

$$\bar{T} = \rho^l \hat{X}(v_1, v_2)\, \hat{Y}(v_3, v_4) \tag{3.8}$$

where $\hat{X}$, $\hat{Y}$ are similar but not identical to X, Y in (3.3). From (2.36) the functions V and $\bar{V}$ differ by the equivalence transformation factors only. If we take the product of these factors over all cubes in a layer, then the result can be put in the form $L(\phi)/L(\phi')$, where ϕ is the set of spins on the lower layer, ϕ' on the upper.

We have therefore shown that

$$[\hat{X}(v_1, v_2)\, \hat{Y}(v_3, v_4)]_{\phi,\phi'} = L(\phi)[X(v_3, v_4)\, Y(v_1, v_2)]_{\phi,\phi'}/L(\phi') \tag{3.9}$$

or in matrix notation

$$\hat{X}(v_1, v_2)\, \hat{Y}(v_3, v_4) = LX(v_3, v_4)\, Y(v_1, v_2)\, L^{-1} \tag{3.10}$$

where L is the diagonal matrix with elements $L(\phi)\,\delta(\phi, \phi')$. It depends on θ_1, but not on any other parameters.

Now introduce the constraints (3.6). This is equivalent to introducing a factor

$$\prod \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \omega^{k(\sigma + \sigma' + \sigma'' + \cdots)} \right\} \tag{3.11}$$

into the summand in (3.1), where σ, σ', σ'',... are the n center spins in a horizontal front-to-back line, and the outer product is over l/n such lines in the horizontal layer of spins ϕ''.

Let V_k be a function that differs from V only in that an extra single spin factor $\omega^{k\sigma}$ is put into the summand in (1.12). Inserting (3.11) into (3.1), we obtain

$$T = \prod \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \prod V_k \right\} \tag{3.12}$$

where the inner product is over n cubes in a horizontal front-to-back line and again the outer product is over all l/n such lines in a layer. From (1.2)

$$V_k(s_0\,|\bar{s}_1, \bar{s}_2, \bar{s}_3\,|\,s_1, s_2, s_3|\,\bar{s}_0) = V(s_0 + k\,|\bar{s}_1, \bar{s}_2, \bar{s}_3 + k\,|\,s_1, s_2, s_3|\,\bar{s}_0) \tag{3.13}$$

Shifting s_0 and $\bar{s}_3$ in (2.36), we obtain

$$\begin{aligned} \frac{V_k}{\bar{V}_k} &= \frac{\gamma(k, s_0 - \bar{s}_3)\, w(v_5, \bar{s}_1 + \bar{s}_2 - s_0 - s_3 - k)}{\gamma(k, \bar{s}_0 - s_3)\, w(v_5, s_1 + s_2 - \bar{s}_0 - \bar{s}_3 - k)} \\ &\quad \times \frac{w(v_5, s_1 + s_2 - \bar{s}_0 - \bar{s}_3)}{w(v_5, \bar{s}_1 + \bar{s}_2 - s_0 - s_3)} \frac{V}{\bar{V}} \end{aligned} \tag{3.14}$$

where $\bar{V}_k = V_k(\bar{s}_{0123})$. If we take the product of each side of (3.14) over the n cubes in a horizontal front-to-back line, the w and s factors cancel. It follows that we still have relations (3.10) in the modified model.[7]

Now consider the case when $v_2 = v_3 = 1$. From (3.5) it follows that

$$v_1 = \frac{1}{\omega v_4} = \omega^{-1/2} \left(\tan \frac{\theta_1}{2} \right)^2 \tag{3.15}$$

[7] Applying similar arguments to the product of the weight function V along one front-to-back line of the cube, rather that to the whole horizontal layer, one obtains precisely the two-dimensional star–star relation of the $sl(n)$-chiral Potts model [Eq. (3.19) of ref. 11].

while (1.12) gives

$$V(s_0,\ldots,\bar{s}_0)=\frac{\lambda_1\,\delta(\bar{s}_2-s_1,\,s_3-\bar{s}_0)}{w(v_1,\,\bar{s}_3-s_0-s_1+s_2)\,w(v_4,\,\bar{s}_1-s_2-s_3+\bar{s}_0)} \tag{3.16}$$

where λ_1 is a scalar factor. [From (2.17) λ_1 is infinite, but this is just a feature of the normalization (2.17), and can readily be removed, leaving the following arguments intact.] The function V_k contains an extra $\omega^{k(s_0-\bar{s}_3)}$ factor, but this cancels out of the product in (3.12), so in this case

$$T_{\phi,\phi'}=\prod_{\text{cubes}} V(s_0,\ldots,\bar{s}_0) \tag{3.17}$$

where V is given by (3.16) and the product is over all cubes in a layer. Substituting (3.16) into (3.17) and taking into account (1.18), (2.30), one obtains

$$T_{\phi,\phi'}=\lambda_2\prod_{\text{cubes}}\delta(\bar{s}_2-s_1,\,s_3-\bar{s}_0) \tag{3.18}$$

where λ_2 is a constant.

The product of Kronecker delta functions is zero unless $\phi=\phi'$, or if ϕ, ϕ' differ only by overall shifts of spins on some horizontal front-to-back lines. Restricting attention to the subspace invariant under these shifts, we then obtain from (3.3), (3.18)

$$X(1,\,v_4)\,Y(v_1,\,1)=\lambda I \tag{3.19}$$

where λ is another (nonzero) constant, I is the unit matrix, and v_1 and v_4 are related by (3.15). For the modified model the matrices in (3.18) are square and hence nonsingular and invertible.[8]

Regard θ_1 as fixed, v_1 and v_3 as independent variables, and v_2 and v_4 as given by (3.5). Then we can suppress the θ_1, v_2, v_4 dependence and write (3.10) as

$$\hat{X}(v_1)\,\hat{Y}(v_3)=LX(v_3)\,Y(v_1)\,L^{-1} \tag{3.20}$$

Then (3.19) gives

$$X(1)\,Y(x)=\hat{X}(x)\,\hat{Y}(1)=\lambda I \tag{3.21}$$

where $x=\omega^{-1/2}(\tan\,\theta_1/2)^{2/N}$ is the value of v_1 when $v_2=1$.

[8] Curiously enough, it is this apparently innocuous statement that fails for the original model and is the reason for introducing the constraints (3.6).

Using (3.20), (3.21), we can now perform the following transformations:

$$\begin{aligned}
&X(v_3)\,Y(x)\,X(1)\,Y(v_1)\\
&\quad=\lambda X(v_3)\,Y(v_1)=\lambda L^{-1}\hat{X}(v_1)\,\hat{Y}(v_3)\,L\\
&\quad=L^{-1}\hat{X}(v_1)\,\hat{Y}(1)\,\hat{X}(x)\,\hat{Y}(v_3)\,L=X(1)\,Y(v_1)\,X(v_3)\,Y(x) \qquad (3.22)
\end{aligned}$$

Hence the matrix products $X(1)\,Y(v_1)$ and $X(v_3)\,Y(x)$ commute for all values of v_1 and v_3. Assuming that these commuting matrix products can be simultaneously diagonalized, it follows that there must exist diagonal matrices $A(v_3)$ and $B(v_1)$ and a nonsingular matrix P (independent of the variables v_1 and v_3) such that

$$X(v_3)\,Y(x)=PA(v_3)\,P^{-1}, \qquad \lambda^{-1}X(1)\,Y(v_1)=PB(v_1)\,P^{-1} \qquad (3.23)$$

Setting $Q=Y(x)\,P$ and remembering that all matrices depend implicitly on θ_1, we finally obtain

$$\begin{aligned}
X(v_3, v_4)&=P(\theta_1)\,A(v_3, v_4)\,Q^{-1}(\theta_1)\\
Y(v_1, v_2)&=Q(\theta_1)\,B(v_1, v_2)\,P^{-1}(\theta_1)
\end{aligned} \qquad (3.24)$$

Here all the matrices depend on θ_1, θ_2, θ_3 only via the arguments explicitly shown.

Note that (3.24) imply the commutativity relation (3.4) for the modified model. [Using then simple arguments like those at the end of Section 2 of ref. 10 we can easily extend (3.4) to the original model as well.]

Thus, the relations (1.15)–(1.20) imply the star–star relation (2.36), which in turn implies (3.20) and the relation (3.24).

From (1.2), (3.3), and (3.24) it follows that

$$\begin{aligned}
Z&=\rho^M\,\mathrm{Tr}(A(v_3, v_4)\,B(v_1, v_2))^m\\
&=\rho^M\sum_j\,(A_{jj}(v_3, v_4)\,B_{jj}(v_1, v_2))^m
\end{aligned}$$

where the j-summation is over all the diagonal elements A_{jj} and B_{jj} of A and B. Assuming that the largest term in the summation is unique and writing a^l for A_{00} and b^l for B_{00} (choosing $j=0$ for the largest term), it follows from (3.25), (2.25) that

$$\kappa=\rho a(v_3, v_4)\,b(v_1, v_2) \qquad (3.25)$$

where ρ is given by (2.23). This factorization property will be used in the next section.

3.2. Partition Function per Site

If $\mathscr{L}$ is infinite in all directions, we can evaluate the partition function per site solely from the symmetry and the factorization properties. In fact, the definition (1.1) is invariant under the symmetry transformation of the lattice generated by the symmetry group of the cube $\mathscr{G}(\mathscr{C})$. On the other hand, from (2.31) and (2.34) the effect of any such transformation on the weight function V is to multiply it by some equivalence transformation factors and permute α_0, α_1, α_2, α_3. Since two such transformations in (2.31), (2.34) generate all possible permutations of α_0, α_1, α_2, α_3, the partition function obeys the following symmetry relation:

$$\kappa(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \kappa(\alpha_i, \alpha_j, \alpha_k, \alpha_l) \tag{3.26}$$

where (i, j, k, l) is any permutation of $(0, 1, 2, 3)$.

Thus we have two functional equations for κ, the factorization property (3.25), (2.23) and the symmetry property (3.26). We shall see that these two functional equations define κ up to a multiplicative constant which is determined then from the normalization (2.29).

The partition function per site κ obviously depends on the number of the spin states N. We can exhibit this by writing it as κ_N. Let us concentrate on the N dependence in the functional equations. The symmetry relation (3.26) does not involve N at all. However, there are two possible courses of the N dependence in the factorization equation (3.25). First, ρ in (3.25) depends on N as given by (2.23). Second, the formulas (2.13) expressing the variables $v_1,..., v_4$ through the angles θ_1, θ_2, θ_3 have the N dependence coming from (2.11). This, however, can be readily removed, since the Nth powers of the variables $v_1,..., v_4$ are the same for any N [the Nth powers of the RHS of (2.13) do not have any N dependence]. So, redefining the functions a and b, we can rewrite (2.25) in the form

$$\kappa = \rho \bar{a}(x_3, x_4)\, \bar{b}(x_1, x_2) \tag{3.27}$$

where $x_i = v_i^N$, $i = 1,..., 4$,

$$\begin{aligned} x_1 &= e^{i\alpha_3} \tan(\theta_1/2) \tan(\theta_2/2), & x_2 &= -e^{i\alpha_3} \cot(\theta_1/2) \tan(\theta_2/2) \\ x_3 &= e^{-i\alpha_3} \tan(\theta_1/2) \tan(\theta_2/2), & x_4 &= -e^{-i\alpha_3} \cot(\theta_1/2) \tan(\theta_2/2) \end{aligned} \tag{3.28}$$

are expressed merely through the angles $\theta_1, \theta_2, \theta_3$ and the only N dependence in the RHS of (3.27) comes from the factor ρ. Writing κ_N in the form

$$\kappa_N = \lambda_N(\psi)^{2(N-1)/N} \tag{3.29}$$

where λ_N is a numerical constant, and substituting it into (3.27), we get the factorization equation for ψ,

$$\psi = (2\xi)\,\bar{\bar{a}}(x_3, x_4)\,\bar{\bar{b}}(x_1, x_2) \tag{3.30}$$

where ξ is given by (2.24). The symmetry relation for ψ remains, clearly, the same as (3.26). Comparing (3.29) with Eq. (7.33) of ref. 18, we see that ψ satisfies the same factorization and symmetry relations as those for the partition function per site of the Zamolodchikov model, κ_2 (which is the $N=2$ case of the considered model). It was shown in ref. 18 that these relations determine κ_2 to within a numerical constant, so we can conclude that

$$\psi = \lambda\kappa_2 \tag{3.31}$$

If we define the function $G(\beta)$ by

$$\log G(\beta) = \frac{1}{2\pi}\int_0^\beta \left[x \cot x + \frac{\pi}{2}\tan\frac{x}{2} - \log(2\sin x) \right] dx \tag{3.32}$$

then we can write the result of ref. 18 as

$$\kappa_2 = \tfrac{1}{2} G(\beta_0)\, G(\beta_1)\, G(\beta_2)\, G(\beta_3) \tag{3.33}$$

where $\beta_0, \beta_1, \beta_2, \beta_3$ are given by (2.10). Note that (3.33) satisfies the normalization (2.29). Taking this into account and substituting (3.31) into (3.29) and then into (2.29), we obtain

$$\log \kappa_N(\theta_1, \theta_2, \theta_3) = \frac{2(N-1)}{N} \log \kappa_2(\theta_1, \theta_2, \theta_3) \tag{3.34}$$

which seems to be very simple and interesting result.

4. FREE BOSON MODEL

It is interesting to learn whether the method described above is unique to the considered model or can be applied to the other models as well. In other words, how many solutions are there for the relations (1.15)–(1.20)? We do not know an answer to this question, but can give one more example. It is the free boson (Gaussian) model on the three-dimensional lattice.

Let now the spins at the sites of the lattice take continuous real values $-\infty < s < \infty$ and the summation over each spin s be replaced by the integration $\int_{-\infty}^{\infty} \cdots ds$. With these modifications the definitions (1.1), (1.12) remain valid, but the weights w and γ in (1.12) are now given by

$$w(v, a)=\exp\left(\frac{-ia^2 v}{2(1-v)}\right), \quad \gamma(a, b)=\exp(-iab), \quad \Phi(a)=\exp\left(\frac{-ia^2}{2}\right) \tag{4.1}$$

Obviously,

$$w(v, 0)=1 \tag{4.2}$$

If we define $\tilde{v}$ and $\Delta(v)$ as

$$\tilde{v}=1/v, \qquad \Delta(v)=1-v \tag{4.3}$$

for any v, then the relations (1.15), (1.17), (1.18) remain valid, while (1.16) is replaced with

$$\int_{-\infty}^{\infty} \gamma(a, b)\, \gamma(-b, c)\, db=2\pi\delta(a-c) \tag{4.4}$$

Further, if we also replace the spin summation by the integration in (1.19)–(1.21), then these formulas become valid as well [provided one formally sets $N=1$ in (1.4), (1.6), i.e., $\Delta(v)$ is now given by (4.3) and $\omega=1$]. The scalar functions ϕ_1 and ϕ_2 are now given by

$$\begin{aligned} \phi_1(v_1, v_2)&=\left[\frac{v_2-v_1}{2\pi i(1-v_1)(1-v_2)}\right]^{-1/2} \\ \phi_2(v_3, v_4)&=\left[\frac{v_3-v_4}{2\pi i(1-v_3)(1-v_4)}\right]^{-1/2} \end{aligned} \tag{4.5}$$

Let us parametrize $v_1,\ldots, v_4$ through the angles $\theta_1, \theta_2, \theta_3$, setting $v_i=x_i$, $i=1,\ldots, 4$, where the x's are given by (3.28). Then, in particular, we have [cf. (2.22)]

$$\phi_1(v_1, v_2)\, \phi_2(v_3, v_4)=2\pi\left(\frac{\sin\theta_2}{\sin\theta_3}\right)^{-1} \tag{4.6}$$

Choose the normalization factor ρ in (1.12) as [cf. (2.23)]

$$\rho=(2\xi)^{-2}/(2\pi) \tag{4.7}$$

where ξ is given by (2.24). Finally, the RHS of (2.28) is replaced by $\delta(s_0-s_2-\bar{s}_2+\bar{s}_0)$ and we still have (2.29).

We have now all the required relations to reproduce the results (3.26), (3.27) for the free boson model and calculate its partition function per site κ_B. In fact, all the calculations are identical to those we did before, except that the exponents in (4.7) and (2.23) are different and the factors N in (1.16) and (2.22) are replaced by 2π in (4.4), (4.6). (These factors mutually cancel and therefore have no effect on the final result.) Taking into account these minor differences, we obtain

$$\log \kappa_B = -2 \log \kappa_2 \tag{4.8}$$

A nice feature of the Gaussian model is that its partition function can be found by the elementary "Gaussian integration" directly from the definition (1.1). Together with (3.33), this means that the partition function per site of the N-state model κ_N can be expressed as a *rational* power of a free boson determinant. We see no obvious reason for this fact and it would be quite interesting to learn why it is so. We postpone the evaluation and the analysis of this determinant to a separate publication.

ACKNOWLEDGMENTS

We are indebted to Prof. L. G. Kovács for very useful discussions. When the manuscript of this paper was in preparation we received a preprint[20] partially overlapping with our Section 2. The authors of that paper find a neat form of the symmetry relations for the Boltzmann weight function of the elementary cube absorbing all the equivalence transformation factors into the definition of the weights (however, they do not calculate the normalization factors in these relations). Moreover, that paper contains a discussion of the tetrahedron relation in the model. We thank the authors of ref. 20 for sending a preprint of their paper.

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